

# THE MAGNETIC RAY TRANSFORM ON SURFACES OF NEGATIVE MAGNETIC CURVATURE

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## Abstract:

Given a closed, oriented, Riemannian 2-manifold equipped with an Anosov magnetic flow such that the magnetic curvature is everywhere non-positive, we show that the magnetic ray transform on tensor fields is injective modulo the natural obstruction.

## 1. INTRODUCTION

Let  $(M, g)$  be a closed, oriented Riemannian manifold. Consider the function  $H : TM \rightarrow \mathbb{R}$  given by

$$H(x, v) := \frac{1}{2}g(v, v), \quad (x, v) \in TM.$$

The geodesic flow on  $TM$  is given by the Hamiltonian flow of the above function with reference to the symplectic structure  $\omega_0$  on  $TM$  provided by the pullback, via the metric, of the canonical symplectic form on  $T^*M$ . The abstract formulation of a magnetic field imposed on  $M$  is specified by a closed 2-form  $\Omega$ . The *magnetic flow*, or twisted geodesic flow, is defined as the Hamiltonian flow of  $H$  under the symplectic form  $\omega$ , where

$$\omega := \omega_0 + \pi^*\Omega,$$

and  $\pi : TM \rightarrow M$  is the usual projection. Magnetic flows were first studied in [An2, Ar]; for more recent references in relation to inverse problems, see below.

We may alternatively think of the magnetic field as being determined by the unique bundle map  $Y : TM \rightarrow TM$ , defined via,

$$\Omega_x(\xi, \eta) = g(Y_x(\xi), \eta), \quad \forall x \in M, \forall \xi, \eta \in T_x M.$$

Note that this implies that  $Y$  is skew-symmetric. The advantage of this point of view is that it provides a nice description of the generator of the magnetic flow, indeed, one can show that this vector field at  $(x, v) \in TM$  is given by

$$X(x, v) + Y_k^i(x)v^i \frac{\partial}{\partial v^k}.$$

Here note that the coefficients of  $Y$  are given by  $Y(\frac{\partial}{\partial x^j}) = Y_i^j \frac{\partial}{\partial x^i}$ , and  $X$  denotes the geodesic vector field. Integral curves of the magnetic flow preserve  $H$ , and thus have constant speed. In what follows we will restrict ourselves to working on the unit tangent bundle:  $SM := H^{-1}(\frac{1}{2})$ . This is not a genuine restriction from a dynamical point of view, since other energy levels may be understood by simply changing  $\Omega$  to  $c\Omega$ , where

$c \in \mathbb{R}$ . Furthermore, the magnetic geodesics, that is the projection of the integral curves of the magnetic flow to  $M$ , are precisely the solutions  $t \mapsto (\gamma(t), \dot{\gamma}(t))$  to the following equation:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Y(\dot{\gamma}).$$

For oriented surfaces  $SM$  is an  $S^1$ -fibration with a circle action on the fibres inducing a vector field which we shall denote by  $V$ . It is a routine exercise to show that for surfaces the generator of the magnetic flow in fact simplifies to:  $X + \lambda V$ , where  $\lambda \in C^\infty(M)$  is the unique function satisfying:  $\Omega = \lambda d\text{vol}_g$  for  $d\text{vol}_g$  the area form of  $M$ . In this article we will proceed under the assumption that  $M$  is a surface.

Given a magnetic flow  $\varphi_t$  on  $M$ , denote by  $\mathcal{G}(M, g, \Omega)$  the set of periodic orbits. We may define the *magnetic ray transform* of a function  $f \in C^\infty(SM)$  by  $I(f) : \mathcal{G}(M, g, \Omega) \rightarrow \mathbb{R}$

$$I(f)(\gamma) := \int_0^T f(\gamma(t), \dot{\gamma}(t)) dt, \quad \gamma \in \mathcal{G}(M, g, \Omega) \text{ has period } T.$$

Now one is lead to ask about the injectivity of this mapping. That is, assume  $I(f)(\gamma) = 0$  for all closed orbits  $\gamma$  of the magnetic flow, is  $f \equiv 0$ ? Clearly, if one takes  $f = (X + \lambda V)u$  for  $u \in C^\infty(SM)$ , then  $I(f) \equiv 0$ . Hence, there is a natural obstruction to injectivity, however, the question remains: are these the only elements in the kernel? In order to characterize the kernel of the ray transform so succinctly one would expect to have to impose some condition on the flow itself, so that the space of closed orbits is sufficiently rich. To this end we stipulate that our flow is Anosov. This means that there exists a continuous splitting  $T(SM) = E^0 \oplus E^u \oplus E^s$  where  $E^0$  is the flow direction, and there are constants  $C > 0$  and  $0 < \rho < 1 < \eta$  such that for all  $t > 0$  we have

$$\|d\varphi_{-t}|_{E^u}\| \leq C\eta^{-t} \quad \text{and} \quad \|d\varphi_t|_{E^s}\| \leq C\rho^t$$

It is shown in [An1] that given a measure preserving Anosov flow, its periodic orbits are dense in the space of all orbits. (Magnetic flows on  $SM$  preserve the Liouville measure, induced from the volume form specified below.) The smooth Livsic theorem [Ll] indeed shows that given a transitive, Anosov flow  $I(f) = 0$  iff there exists  $u \in C^\infty(SM)$  such that  $(X + \lambda V)u = f$ . (Since the magnetic flow is volume preserving, its non-wandering set is all of  $SM$ , therefore if the flow is, in addition, Anosov, then it must be transitive.) Now we wish to ask a more refined question about the kernel. In order to do so we need to digress to introduce some Fourier analysis.

As previously let  $X$  denote the vector field on  $SM$  generated by the geodesic flow, and  $V$  the vector field induced by the circle action on the fibres. By defining  $X_\perp := [X, V]$ , we obtain a global frame  $\{X, X_\perp, V\}$  for  $T(SM)$ . The two remaining commutators will play an important role in what follows, and are given by (see [Si]):  $[V, X_\perp] = X$ , and  $[X, X_\perp] = -KV$ , where  $K$  is the Gaussian curvature of  $M$ . We define a Riemannian metric on  $SM$  by declaring that  $\{X, X_\perp, V\}$  form an orthonormal basis, and will denote by  $d\Sigma^3$  the volume form of this metric.

We define an inner product between functions  $u, v : SM \rightarrow \mathbb{C}^n$  as follows:

$$\langle u, v \rangle := \int_{SM} \langle u, v \rangle_{\mathbb{C}^n} d\Sigma^3.$$

Now,  $L^2(SM, \mathbb{C}^n)$  decomposes orthogonally as

$$L^2(SM, \mathbb{C}^n) = \bigoplus_{k \in \mathbb{Z}} H_k$$

where  $-iV$  acts as  $k\text{Id}$  on  $H_k$ . Thus, we can decompose a smooth function  $u : SM \rightarrow \mathbb{C}^n$  into its Fourier components

$$u = \sum_{k=-\infty}^{\infty} u_k$$

where  $u_k \in \Omega_k := C^\infty(SM, \mathbb{C}^n) \cap H_k$ .

Given any symmetric, covariant  $m$ -tensor field  $f = f_{i_1 \dots i_m} dx^{i_1} \otimes \dots \otimes dx^{i_m}$  on  $M$  we can associate a function  $\hat{f}$  on  $SM$  by

$$\hat{f}(x, v) := f_{i_1 \dots i_m} v^{i_1} \dots v^{i_m}.$$

Now such a  $\hat{f}$  can be decomposed into its Fourier components as  $\hat{f} = \sum_{k=-m}^m \hat{f}_k$ , and in general if any function on  $SM$  has nontrivial Fourier components only for  $-m \leq k \leq m$ , then we say that it is of degree  $m$ . In what follows we will drop the hat leaving it clear from the context when we mean  $f$  to induce a function on  $SM$ .

The tensor tomography problem is to determine the kernel of the ray transform when it acts on functions on  $SM$  which are induced by tensor fields. Let us assume at first that  $\Omega = 0$ , so that we are working with the standard geodesic ray transform. Given our setup the Livsic theorem implies that for any symmetric  $m$ -tensor  $f$  satisfying  $I(f) = 0$ , there exists  $u \in C^\infty(SM)$  such that  $Xu = f$ . The right hand side has degree  $m$ . Does this imply that  $u$  has degree  $m - 1$ ? (When  $f$  has degree 0 we interpret the question to be: does  $f \equiv 0$ ?) This inverse question has received considerable attention recently. It is well known that the geodesic flow of a negatively curved  $n$ -manifold is Anosov (though the converse doesn't hold), moreover in [Gu] it is shown that on a closed, oriented, negatively curved surface our question is resolved affirmatively, and furthermore in [Cr] that on a closed, oriented, negatively curved  $n$ -manifold, the same result holds. Ideally, one would like to extend these results to the general Anosov case, and remove the curvature assumption. Partial results in this direction were achieved in [Da6]. In particular, for the Anosov case for surfaces it is shown there that the statement is true for tensors of rank  $m = 0, 1$ , and in [Sh2] it's shown that it holds for  $m = 2$  with the additional assumption that  $(M, g)$  has no focal points.

Now let us proceed to the general magnetic case where  $\Omega$  is arbitrary. In the magnetic setting, the flow couples components of differing degrees, hence the analogous question requires us to consider sums of functions induced by tensors of differing ranks. Therefore, for each  $0 \leq i \leq m$  let  $f_i$  be a symmetric  $i$ -tensor, inducing a function on  $SM$ , and consider  $f = f_0 + \dots + f_m$ . Given our setup the Livsic theorem implies that if  $f$  satisfies  $I(f) = 0$ , there exists  $u \in C^\infty(SM)$  such that  $(X + \lambda V)u = f$ . The right hand side has degree  $m$ . Does this imply that  $u$  has degree  $m - 1$ ? (When  $f$  has degree 0 we interpret the question to be: does  $f \equiv 0$ ?)

In the general magnetic case most of the results hithertofore are only for tensors up to degree 1. In [Da4] the above question for surfaces with an Anosov magnetic flow is resolved affirmatively for tensors up to degree 1. In [Da2] the same statement is proved, but the more general class of Anosov thermostat flows are considered, and in [Ja] some further partial results for thermostats are achieved. In [Da3] it is shown that the Riemannian hypothesis can be weakened to Finsler, and the statement proven for tensors up to degree 1 on manifolds of arbitrary dimension. In [Da5] positive results are obtained even when the flow is not Anosov, but simply has no conjugate points.

From the above discussion in the geodesic case, one might hope for results in the case where one has an Anosov magnetic flow and negative Gaussian curvature (since in the magnetic setting negative curvature alone is not sufficient to guarantee that the flow is Anosov). It turns out that the appropriate quantity to consider in the magnetic setting is magnetic curvature, which we follow [Bu] in defining to be:  $K + X_\perp(\lambda) + \lambda^2$ . Now if  $(M, g, \Omega)$  has negative magnetic curvature, then the magnetic flow is Anosov [Wo], in analogy with

the geodesic case. This leads us to the main result of this paper which resolves a question initially posed in [Ja]. There the problem was only stated for 2-tensors, yet here we managed to solve it for tensors of arbitrary rank.

**THEOREM: 1.1.** *Let  $(M, g, \Omega)$  be a closed, oriented Riemannian 2-manifold equipped with an Anosov magnetic flow. Suppose that  $K + X_\perp(\lambda) + \lambda^2 \leq 0$ . Let  $f = f_0 + \dots + f_m$  where  $f_i$  is a symmetric  $i$ -tensor. If  $I(f) = 0$ , then  $f = (X + \lambda V)a$  where  $a \in C^\infty(SM)$  is of degree  $m - 1$ .*

Note that for the proof we need only consider when the genus of  $M$  is  $\geq 2$ . This is a consequence of the fact that the fundamental group of any  $S^1$  bundle over the 2-sphere or torus has polynomial growth, and a classic result of Plante and Thurston [Pl] which says that if an  $S^1$ -fibration over a surface carries an Anosov flow, then the fundamental group of the fibration must grow exponentially.

We will exploit the following lemma to achieve our result. Its proof is given in [Pa3].

**LEMMA: 1.2.** *Assume  $(M, g)$  has genus  $\geq 2$ . Then  $\eta_+ : \Omega_k \rightarrow \Omega_{k+1}$  is injective for  $k \geq 1$ , and  $\eta_- : \Omega_k \rightarrow \Omega_{k-1}$  is injective for  $k \leq -1$ . Here  $\eta_+ := \frac{1}{2}(X + iX_\perp)$  and  $\eta_- := \frac{1}{2}(X - iX_\perp)$ .*

The other crucial component in our proof is the Pestov Identity (see [Da2] for a succinct proof) which has recently been used in various guises in the resolution of inverse problems, see for example [Da3, Pa1, Pa2].

**THEOREM: 1.3 (PESTOV'S IDENTITY).** *Let  $(M, g, \Omega)$  be a closed, oriented Riemannian 2-manifold equipped with a magnetic flow. If  $u \in C^\infty(SM)$  then the following holds:*

$$\|V(X + \lambda V)u\|^2 = \|(X + \lambda V)Vu\|^2 - ((K + X_\perp(\lambda) + \lambda^2)Vu, Vu) + \|(X + \lambda V)u\|^2.$$

Throughout this paper we are restricting ourselves to the case where  $M$  is closed, however, one may ask analogous questions when  $M$  is compact with non-empty boundary, see for example [Ai, Da1, Pa1, Pa2].

## 2. PROOF OF THE MAIN RESULT

**DEFINITION: 2.1.** *Let  $\alpha \in [0, 1]$ . We say that  $(M, g, \Omega)$  is  $\alpha$ -controlled if*

$$\|(X + \lambda V)u\|^2 - ((K + X_\perp(\lambda) + \lambda^2)u, u) \geq \alpha \|(X + \lambda V)u\|^2$$

for all  $u \in C^\infty(SM)$ .

In the proofs that follow we will denote by  $T : C^\infty(SM) \rightarrow \bigoplus_{|k| \geq m+1} \Omega_k$  the projection operator, defined by:

$$Tu = \sum_{|k| \geq m+1} u_k.$$

**PROPOSITION: 2.2.** *Suppose that the triple  $(M, g, \Omega)$  is  $\alpha$ -controlled and let  $m$  be an integer  $\geq 1$ . Then given any  $u \in \bigoplus_{|k| \geq m} \Omega_k$  we have*

$$\|TV(X + \lambda V)u\|^2 \geq (1 - (m-1)^2 + \alpha m^2)(\|\eta_- u_m\|^2 + \|\eta_+ u_{-m}\|^2) + (1 - m^2 + \alpha(m+1)^2)(C^+ + C^-) + \alpha \|w\|^2 + \|v\|^2$$

where  $v := \sum_{|k| \geq m+1} ((X + \lambda V)u)_k$ ,  $C^+ := \min \left\{ \|im\lambda u_m + \eta_- u_{m+1}\|^2, \left\| \frac{im^2}{m+1} \lambda u_m + \eta_- u_{m+1} \right\|^2 \right\}$ ,

$w := \sum_{|k| \geq m+1} ((X + \lambda V)Vu)_k$ , and  $C^- := \min \left\{ \|i(-m)\lambda u_{-m} + \eta_+ u_{-(m+1)}\|^2, \left\| \frac{im^2}{m+1} \lambda u_{-m} - \eta_+ u_{-(m+1)} \right\|^2 \right\}$ .

*Proof.* Given  $u \in \bigoplus_{|k| \geq m} \Omega_k$  we compute:

$$\begin{aligned}
\sum_{|k| \leq m} k^2 \|((X + \lambda V)u)_k\|^2 &= (m-1)^2 \|\eta_- u_m\|^2 + m^2 \|im\lambda u_m + \eta_- u_{m+1}\|^2 + (m-1)^2 \|\eta_+ u_{-m}\|^2 \\
&\quad + m^2 \|-im\lambda u_{-m} + \eta_+ u_{-(m+1)}\|^2. \\
\|(X + \lambda V)u\|^2 &= \|im\lambda u_m + \eta_- u_{m+1}\|^2 + \|\eta_- u_m\|^2 + \|i(-m)\lambda u_{-m} + \eta_+ u_{-(m+1)}\|^2 + \|\eta_+ u_{-m}\|^2 \\
&\quad + \left\| \sum_{|k| \geq m+1} ((X + \lambda V)u)_k \right\|^2. \\
\|(X + \lambda V)Vu\|^2 &= \|-m^2\lambda u_m + \eta_- i(m+1)u_{m+1}\|^2 + \|\eta_- mu_m\|^2 + \|-m^2\lambda u_{-m} - \eta_+ i(m+1)u_{-(m+1)}\|^2 \\
&\quad + \|\eta_+ mu_{-m}\|^2 + \left\| \sum_{|k| \geq m+1} ((X + \lambda V)Vu)_k \right\|^2. \\
\|V(X + \lambda V)u\|^2 &= \sum_{|k| \leq m} k^2 \|((X + \lambda V)u)_k\|^2 + \|TV(X + \lambda V)u\|^2 \\
&= (m-1)^2 \|\eta_- u_m\|^2 + m^2 \|im\lambda u_m + \eta_- u_{m+1}\|^2 + (m-1)^2 \|\eta_+ u_{-m}\|^2 \\
&\quad + m^2 \|-im\lambda u_{-m} + \eta_+ u_{-(m+1)}\|^2 + \|TV(X + \lambda V)u\|^2. \tag{1}
\end{aligned}$$

Now we use Pestov's Identity and our hypothesis:

$$\begin{aligned}
&\|V(X + \lambda V)u\|^2 \\
&= \|(X + \lambda V)Vu\|^2 - ((K + X_\perp(\lambda) + \lambda^2)Vu, Vu) + \|(X + \lambda V)u\|^2 \\
&\geq \alpha \|V(X + \lambda V)u\|^2 + \|(X + \lambda V)u\|^2 \\
&= \alpha \|-m^2\lambda u_m + \eta_- i(m+1)u_{m+1}\|^2 + \alpha \|\eta_- mu_m\|^2 \\
&\quad + \alpha \|-m^2\lambda u_{-m} - \eta_+ i(m+1)u_{-(m+1)}\|^2 + \alpha \|\eta_+ mu_{-m}\|^2 + \alpha \|w\|^2 \\
&\quad + \|im\lambda u_m + \eta_- u_{m+1}\|^2 + \|\eta_- u_m\|^2 + \|i(-m)\lambda u_{-m} + \eta_+ u_{-(m+1)}\|^2 + \|\eta_+ u_{-m}\|^2 + \|v\|^2.
\end{aligned}$$

To conclude we simply combine this inequality with equation (1). □

*Proof of Theorem 1.1.* From the comments in the introduction it is no restriction to simply consider the case when the genus of  $M$  is  $\geq 2$ . We proceed under this assumption.

The Livsic Theorem [L1] guarantees that  $I(f) = 0$  iff there exists  $a \in C^\infty(SM)$  such that  $(X + \lambda V)a = f$ . Given such a function, define  $u := a - \sum_{|k| \leq m-1} a_k$ . Then  $(X + \lambda V)u$  has degree  $m$ , and  $TV(X + \lambda V)u = 0$ . Now any triple  $(M, g, \Omega)$  with non-positive magnetic curvature is clearly 1-controlled, and thus  $\alpha$ -controlled for all  $\alpha \in [0, 1]$ . In particular, it is  $\frac{m-1}{m+1}$ -controlled. We now apply Proposition 2.2 with  $\alpha = \frac{m-1}{m+1}$  to obtain:

$$(X + \lambda V)u = im\lambda u_m + \eta_- u_{m+1} + i(-m)\lambda u_{-m} + \eta_+ u_{-(m+1)}$$

$$(X + \lambda V)Vu = -m^2\lambda u_m + i(m+1)\eta_-u_{m+1} - m^2\lambda u_{-m} - i(m+1)\eta_+u_{-(m+1)}.$$

Now,

$$\begin{aligned} X_\perp u &= (X + \lambda V)Vu - V(X + \lambda V)u \\ &= -m^2\lambda u_m + i(m+1)\eta_-u_{m+1} - m^2\lambda u_{-m} - i(m+1)\eta_+u_{-(m+1)} \\ &\quad - [-m^2\lambda u_m + im\eta_-u_{m+1} - m^2\lambda u_{-m} - im\eta_+u_{-(m+1)}] \\ &= i\eta_-u_{m+1} - i\eta_+u_{-(m+1)}. \end{aligned}$$

But we also have  $X_\perp u = i\eta_-u - i\eta_+u$ . Therefore,  $\eta_-u = \eta_-u_{m+1}$  and  $\eta_+u = \eta_+u_{-(m+1)}$ . Therefore, Lemma 1.2 allows us to conclude that  $u \equiv 0$ .  $\square$

REMARK: 2.3. The proof we have given above obviously doesn't work when  $f = f_0 \in C^\infty(M)$ , however in this particular case the result is known thanks to [Da2].

We note that the above proof does not require the Anosov property, with the exception of its initial use to invoke the Livsic theorem. In [Da5] a similar result was obtained without assuming Anosov, but instead working with the horocycle flow on a compact hyperbolic surface. The proof there relied on an analogous result to the Livsic theorem, and substantial arguments from representation theory, both of which are derived in [Fl]. We note that by simply employing the analogue of the Livsic theorem, our theorem recovers the result, Proposition 4.3, from [Da5] (except for the  $f = f_0$  case).

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